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**LETTER TO THE EDITOR**

# Quasi-exactly solvable quartic potential

Carl M Bender† and Stefan Boettcher‡§

† Department of Physics, Washington University, St Louis, MO 63130, USA
‡ Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

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**Abstract.** A new two-parameter family of quasi-exactly solvable quartic polynomial potentials $V(x) = -x^4 + 2iax^3 + (a^2 - 2b)x^2 + 2i(ab - J)x$ is introduced. Heretofore, it was believed that the lowest-degree one-dimensional quasi-exactly solvable polynomial potential is sextic. This belief is based on the assumption that the Hamiltonian must be Hermitian. However, it has recently been discovered that there are huge classes of non-Hermitian, $\mathcal{PT}$-symmetric Hamiltonians whose spectra are real, discrete, and bounded below. Replacing hermiticity by the weaker condition of $\mathcal{PT}$ symmetry allows for new kinds of quasi-exactly solvable theories. The spectra of this family of quartic potentials discussed here are also real, discrete and bounded below and the quasi-exact portion of the spectra consists of the lowest $J$ eigenvalues. These eigenvalues are the roots of a $J$th-degree polynomial.

Quantum-mechanical potentials are said to be *quasi-exactly solvable* (QES) if a finite portion of the energy spectrum and associated eigenfunctions can be found exactly and in closed form [1]. QES potentials depend on a parameter $J$; for positive integer values of $J$ one can find exactly the first $J$ eigenvalues and eigenfunctions, typically of a given parity. QES systems can be classified using an algebraic approach in which the Hamiltonian is expressed in terms of the generators of a Lie algebra [2–5]. This approach generalizes the dynamical-symmetry analysis of *exactly solvable* quantum-mechanical systems, whose *entire* spectrum may be found in closed form by algebraic means [6].

An especially simple and well known example of a QES potential [7] is

$$V(x) = x^6 - (4J - 1)x^2. \tag{1}$$

The Schrödinger equation, $-\psi''(x) + [V(x) - E]\psi(x) = 0$, has $J$ even-parity solutions of the form

$$\psi(x) = e^{-x^4/4} \sum_{k=0}^{J-1} c_k x^{2k}. \tag{2}$$

The coefficients $c_k$ for $0 \leqslant k \leqslant J - 1$ satisfy the recursion relation

$$4(J - k)c_{k-1} + Ec_k + 2(k + 1)(2k + 1)c_{k+1} = 0 \tag{3}$$

where we define $c_{-1} = c_J = 0$. The simultaneous linear equations (3) have a nontrivial solution for $c_0, c_1, \ldots, c_{J-1}$ if the determinant of the coefficients vanishes. For each integer $J$ this determinant is a polynomial of degree $J$ in the variable $E$. The roots of this polynomial are all real and are the $J$ quasi-exact energy eigenvalues of the potential (1).

§ Also at: Center for Theoretical Studies of Physical Systems, Clark Atlanta University, Atlanta, GA 30314, USA.

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The lowest-degree one-dimensional QES polynomial potential that is discussed in the literature is sextic. However, in this paper we introduce an entirely new two-parameter class of QES *quartic* polynomial potentials. The spectra of this family of potentials are real, discrete, and bounded below. Like the eigenvalues of the potential (1), the lowest $J$ eigenvalues of these potentials are the roots of a polynomial of degree $J$.

The potentials introduced here have not been discovered so far because they are associated with non-Hermitian Hamiltonians. Recently, it has been found that there are large classes of non-Hermitian Hamiltonians whose spectra are real and bounded below [8, 9]. Although they are non-Hermitian, these Hamiltonians exhibit the weaker symmetry of $\mathcal{PT}$ invariance. A class of these Hamiltonians,

$$H = p^2 - (\mathrm{i}x)^N \qquad (N \geqslant 2) \tag{4}$$

was studied in [8]. The special case $N = 4$ corresponds to the Hamiltonian

$$H = p^2 - x^4. \tag{5}$$

It is not at all obvious that this Hamiltonian has a positive, real, discrete spectrum. To verify this property, we must continue analytically the Schrödinger equation eigenvalue problem associated with $H$ in (4) from the conventional harmonic oscillator ($N = 2$) to the case $N = 4$. In doing so, the boundary conditions at $|x| = \infty$ rotate into the complex $x$ plane. At $N = 4$ the boundary conditions on the wavefunction $\psi(x)$ read

$$\lim_{|x|\to\infty} \psi(x) = 0 \tag{6}$$

where the limit $x \to \infty$ is taken inside two wedges bounded by the Stokes' lines of the differential equation. The right wedge is bounded by the Stokes' lines at $0^\circ$ and $-60^\circ$ and the left wedge is bounded by the Stokes' lines at $-120^\circ$ and $-180^\circ$. The leading asymptotic behaviour of the wavefunction is given by

$$\psi(x) \sim \mathrm{e}^{-\mathrm{i}x^3/3} \qquad (|x| \to \infty). \tag{7}$$

It is easy to see that the asymptotic conditions in (6) are satisfied by $\psi(x)$. A complete discussion of the analytic continuation of eigenvalue problems into the complex plane is given in [10]. Note that for all values of $N$ between 2 and 4, the Hamiltonian (4) is not symmetric under parity. This parity noninvariance persists even at $N = 4$; eigenfunctions $\psi(x)$ of (5) are not symmetric (or antisymmetric) under the replacement $x \to -x$.

In this paper we generalize the Hamiltonian (5) to the two-parameter class

$$H = p^2 - x^4 + 2\mathrm{i}ax^3 + (a^2 - 2b)x^2 + 2\mathrm{i}(ab - J)x \tag{8}$$

where $a$ and $b$ are real and $J$ is a positive integer. The wavefunction $\psi(x)$ satisfies the boundary conditions (6) and the differential equation

$$E\psi(x) = -\psi''(x) + [-x^4 + 2\mathrm{i}ax^3 + (a^2 - 2b)x^2 + 2\mathrm{i}(ab - J)x]\psi(x). \tag{9}$$

We obtain the QES portion of the spectrum of $H$ in (8) as follows. We make the *ansatz*

$$\psi(x) = \mathrm{e}^{-\mathrm{i}x^3/3 - ax^2/2 - \mathrm{i}bx} P_{J-1}(x) \tag{10}$$

where

$$P_{J-1}(x) = x^{J-1} + \sum_{k=0}^{J-2} c_k x^k \tag{11}$$

is a polynomial in $x$ of degree $J - 1$. Substituting $\psi(x)$ in (10) into the differential equation (9), dividing by the exponential in (10), and collecting powers of $x$, we obtain a polynomial in $x$ of degree $J - 1$. Setting the coefficients of $x^k$ ($1 \leqslant k \leqslant J - 1$) to 0 gives a system of $J-1$ simultaneous linear equations for the coefficients $c_k$ ($0 \leqslant k \leqslant J-2$). We solve these equations and substitute the values of $c_k$ into the coefficient of $x^0$. This gives a polynomial $Q_J(E)$ of degree $J$ in the energy eigenvalue $E$. The coefficients of this polynomial are functions of the parameters $a$ and $b$ of the Hamiltonian $H$ in (8). The first five of these polynomials are

$$\begin{aligned}
Q_1 &= E - b^2 - a\\
Q_2 &= E^2 - (2b^2+4a)E + b^4 + 4ab^2 - 4b + 3a^2\\
Q_3 &= E^3 - (3b^2+9a)E^2 + (3b^4+18ab^2-16b+23a^2)E - b^6 - 9ab^4 + 16b^3\\
&\qquad -23a^2b^2 + 48ab - 15a^3 - 16\\
Q_4 &= E^4 - (4b^2+16a)E^3 + (6b^4+48ab^2-40b+86a^2)E^2\\
&\qquad +(-4b^6 - 48ab^4 + 80b^3 - 172a^2b^2 + 320ab - 176a^3 - 96)E\\
&\qquad +b^8 + 16ab^6 - 40b^5 + 86a^2b^4 - 320ab^3 + 176a^3b^2 + 240b^2\\
&\qquad -568a^2b + 105a^4 + 384a\\
Q_5 &= E^5 - (5b^2+25a)E^4 + (10b^4+100ab^2-80b+230a^2)E^3\\
&\qquad +(-10b^6 - 150ab^4 + 240b^3 - 690a^2b^2 + 1200ab - 950a^3 - 336)E^2\\
&\qquad +(5b^8 + 100ab^6 - 240b^5 + 690a^2b^4 - 2400ab^3\\
&\qquad +1900a^3b^2 + 1696b^2 - 5488a^2b + 1689a^4 + 3360a)E\\
&\qquad -b^{10} - 25ab^8 + 80b^7 - 230a^2b^6 + 1200ab^5 - 950a^3b^4 - 1360b^4\\
&\qquad +5488a^2b^3 - 1689a^4b^2 - 8480ab^2 + 7440a^3b + 3072b - 945a^5 - 7632a^2.
\end{aligned} \tag{12}$$

The roots of $Q_J(E)$ are the QES portion of the spectrum of $H$.

The polynomials $Q_J(E)$ simplify dramatically if we substitute

$$E = F + b^2 + Ja \tag{13}$$

and

$$K = 4b + a^2. \tag{14}$$

The new polynomials have the form

$$\begin{aligned}
Q_1 &= F\\
Q_2 &= F^2 - K\\
Q_3 &= F^3 - 4KF - 16\\
Q_4 &= F^4 - 10KF^2 - 96F + 9K^2\\
Q_5 &= F^5 - 20KF^3 - 336F^2 + 64K^2F + 768K\\
Q_6 &= F^6 - 35KF^4 - 896F^3 + 259K^2F^2 + 7040KF - 225K^3 + 25\,600\\
Q_7 &= F^7 - 56KF^5 - 2016F^4 + 784K^2F^3 + 35\,712KF^2 - 2304K^3F\\
&\qquad +288\,000F - 55\,296K^2\\
Q_8 &= F^8 - 84KF^6 - 4032F^5 + 1974K^2F^4 + 132\,480KF^3 - 12\,916K^3F^2\\
&\qquad +1760\,256F^2 - 681\,408K^2F + 11\,025K^4 - 6322\,176K.
\end{aligned} \tag{15}$$

The roots of these polynomials are all real as long as $K \geqslant K_{\rm critical}$, where $K_{\rm critical}$ is a function of $J$. The first few values of $K_{\rm critical}$ are listed in table 1. At $K = K_{\rm critical}$ the lowest two

**Table 1.** Sequence of critical values for $K_{\mathrm{critical}}$ and $F_{\mathrm{critical}}$.

| $J$ | $K_{\mathrm{critical}}$ | $F_{\mathrm{critical}}$ | $J$ | $K_{\mathrm{critical}}$ | $F_{\mathrm{critical}}$ |
|---|---|---|---|---|---|
| 2 | 0.0 | 0.0 | 16 | 25.0526 | −61.3470 |
| 3 | 3.0 | −2.0 | 17 | 26.3475 | −67.3089 |
| 4 | 5.470 86 | −4.718 94 | 18 | 27.6149 | −73.4116 |
| 5 | 7.655 70 | −7.939 82 | 19 | 28.8569 | −79.6490 |
| 6 | 9.651 84 | −11.557 2 | 20 | 30.0754 | −86.0158 |
| 7 | 11.510 4 | −15.507 0 | 21 | 31.2721 | −92.5072 |
| 8 | 13.262 5 | −19.745 9 | 22 | 32.4485 | −99.1187 |
| 9 | 14.928 7 | −24.241 9 | 23 | 33.6058 | −105.846 |
| 10 | 16.523 5 | −28.970 6 | 24 | 34.7453 | −112.686 |
| 11 | 18.057 6 | −33.912 6 | 25 | 35.8679 | −119.635 |
| 12 | 19.539 2 | −39.052 1 | 26 | 36.9747 | −126.689 |
| 13 | 20.974 7 | −44.375 8 | 27 | 38.0665 | −133.846 |
| 14 | 22.369 5 | −49.872 5 | 28 | 39.1439 | −141.103 |
| 15 | 23.727 6 | −55.532 3 | 29 | 40.2078 | −148.458 |

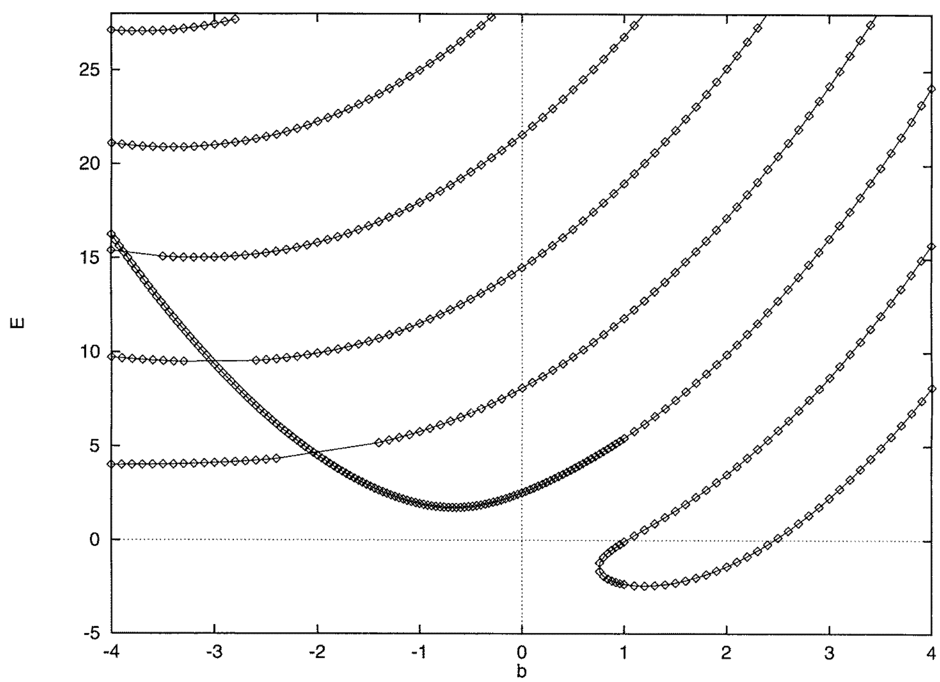


**Figure 1.** The spectrum for the QES Hamiltonian (9) plotted as a function of $b$ for the case $J = 3$ and $a = 0$. For $b > \frac{3}{4}$ (corresponding to the critical value $K_{\mathrm{critical}} = 3$) the QES eigenvalues are real and are the three lowest eigenvalues of the spectrum. When $b$ goes below $\frac{3}{4}$, two of the QES eigenvalues become complex and the third moves into the midst of the non-QES spectrum. We believe that the non-QES spectrum is entirely real throughout the $(a, b)$ plane.

eigenvalues become degenerate and when $K < K_{\mathrm{critical}}$ some of the eigenvalues of the QES spectrum are complex. Thus, the QES spectrum is entirely real above a parabolic-shaped region in the $(a, b)$ plane bounded by the curve $a^2 + 4b = K_{\mathrm{critical}}$.

Extensive numerical calculations lead us to believe that the non-QES spectrum is entirely real throughout the $(a, b)$ plane and that when $K > K_{\mathrm{critical}}$ the eigenvalues of the QES spectrum lie below the eigenvalues of the non-QES spectrum. However, as we enter the region $K < K_{\mathrm{critical}}$ some of the eigenvalues of the QES spectrum pair off and become complex. Other eigenvalues of the QES spectrum may cross above the eigenvalues of the non-QES spectrum. In figure 1 we illustrate the case $J = 3$ and $a = 0$. Note that for $b > \frac{3}{4}$ the QES eigenvalues are three lowest eigenvalues of the spectrum. When $b$ goes below $\frac{3}{4}$, two of the QES eigenvalues become complex and the third moves into the midst of the non-QES spectrum.

The standard way to understand QES theories is to demonstrate that the Hamiltonian can be expressed in terms of generators of a Lie algebra. Following Turbiner [3], we use the generators of a finite-dimensional representation of the $SL(2, Q)$ with spin $J$. The three generators have the form

$$\mathcal{J}^+ = x^2\frac{\mathrm{d}}{\mathrm{d}x} - (J-1)x \qquad \mathcal{J}^0 = x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{J-1}{2}$$
$$\mathcal{J}^- = \frac{\mathrm{d}}{\mathrm{d}x}. \tag{16}$$

If we apply the Hamiltonian $H$ in (8) to $\psi(x)$ in (10) and divide by the exponential we obtain an operator $h$ acting on the polynomial $P_{J-1}(x)$; $h$ has the form

$$h = -\frac{\mathrm{d}}{\mathrm{d}x^2} + (2\mathrm{i}x^2 + 2ax + 2\mathrm{i}b)\frac{\mathrm{d}}{\mathrm{d}x} - [2\mathrm{i}(J-1)x - b^2 - a]. \tag{17}$$

Hence, in terms of the generators of the Lie algebra, we have

$$h = -(\mathcal{J}^-)^2 + 2\mathrm{i}\mathcal{J}^+ + 2a\mathcal{J}^0 + 2\mathrm{i}b\mathcal{J}^- + b^2 + aJ. \tag{18}$$

This algebraic structure possesses $\mathcal{PT}$ symmetry and has *real* eigenvalues.

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